Chapter 5 Appendix

Ω **function.** Theorem 2 is capable of generalization. If f is an additive function then

$$f(n) = \sum_{p^a || n} f(p^a)\,.$$

Thus

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{p^a||n} f(p^a) = \sum_{p^a\le x} f(p^a) \sum_{\substack{n\le x\\ p^a||n}} 1,$$

having interchanged the summations. If $p^a||n$ then $p^a|n$ but $p^{a+1}\nmid n$. So, to count the number of integers $n\le x$ for which $p^a||n$ we count the number with $p^a|n$ and subtract the number with $p^{a+1}|n$. That is,

$$\sum_{\substack{n\le x\\ p^a||n}} 1 = \sum_{\substack{n\le x\\ p^a|n}} 1 - \sum_{\substack{n\le x\\ p^{a+1}|n}} 1 = \left[\frac{x}{p^a}\right] - \left[\frac{x}{p^{a+1}}\right] = \frac{x}{p^a} - \frac{x}{p^{a+1}} + O(1)\,.$$

Therefore

$$\sum_{n\le x} f(n) = x\sum_{p^a\le x}\frac{f(p^a)}{p^a}\left(1-\frac{1}{p}\right) + O\left(\sum_{p^a\le x}|f(p^a)|\right).$$

We see here the main term in the next result.

Theorem 11 ***Turán-Kubilius*** *For any additive function f we have*

$$\sum_{n\le x}\left(f(n)-A(x)\right) \le \left(2+o\left(1\right)\right) xB^2\left(x\right),$$

with

$$A(x) = \sum_{p^n\le x}\frac{f\left(p^n\right)}{p^n}\left(1-\frac{1}{p}\right) \quad \textit{and} \quad B^2(x) = \sum_{p^n\le x}\frac{\left|f\left(p^n\right)\right|^2}{p^n}.$$

Example With $f=\Omega$ it is easily checked that

$$\begin{aligned} A(x) &= \sum_{p^n\le x}\frac{n}{p^n}\left(1-\frac{1}{p}\right) \\ &= \sum_{p\le x}\frac{1}{p} - \sum_{p\le x}\frac{1}{p^2} + \sum_{\substack{p^n\le x\\ n\ge 2}}\frac{n}{p^n}\left(1-\frac{1}{p}\right) \\ &= \log\log x + O\left(1\right), \end{aligned}$$

and

$$B^2(x) = \sum_{p^n \le x} \frac{n^2}{p^n} = \sum_{p \le x} \frac{1}{p} + \sum_{\substack{p^n \le x \\ n \ge 2}} \frac{n^2}{p^n} = \log\log x + O\left(1\right),$$

Thus

Corollary 12

$$\sum_{n \le x} \left(\Omega(n) - \log\log x\right)^2 = O\left(x \log\log x\right).$$

As for the ω function the $\log\log x$ can be replaced by $\log\log n$:

$$\sum_{3 \le n \le x} \left(\Omega(n) - \log\log n\right)^2 = O\left(x \log\log x\right).$$

This leads to the same conclusions but for Ω; $\Omega\left(n\right)$ has normal order $\log\log n$ and almost all integers n have $\log\log n$ prime divisors *counted with multiplicity*.

Probabilistic Number Theory The results in this short chapter were the start of probabilistic Number Theory. An important result of this theory was

Theorem 13 ***Erdos-Kac*** *For all* $\alpha \le \beta$,

$$\lim_{x \to \infty} \frac{1}{x} \left\{ n \le x : \alpha \le \frac{\omega(n) - \text{loglog}n}{\sqrt{\text{loglog}n}} \le \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt.$$

This says that the function

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

is normally distributed (in some sense) with mean $\log\log n$ and standard deviation $\sqrt{\log\log n}$.